

# Results on the solutions of maximum weighted Renyi entropy problems

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## Abstract

In this paper, following standard arguments, the maximum Renyi entropy problem for the weighted case is analyzed. We verify that under some constraints on weight function, the Student- $r$  and Student- $t$  distributions maximize the weighted Renyi entropy. Furthermore, an extended version of the Hadamard inequality is derived.

## 1 Introduction

It is well-known that entropy has been widely played an important role in optimization problems which preserve various applications in areas of computer vision, communication transmission, medical and so on presented in the literature. Thus, studying the entropy maximizing distributions became a principal object in information theory for understanding the Shannon entropy optimization, and later extended versions of problems such as Renyi and Tsallis entropies maximization. See [12, 10, 24, 4, 22, 9].

In 1968-1971, subsequently followed by Shannon entropy concept, the initial definition of *weighted entropy* was illustrated in [2, 7]. Following the weighted progress, recently further results in [3, 16, 17, 18, 19, 20] with a number of theoretical suggestions have been established. As a kind of fundamental reference, in [16] the maximization of weighted entropy and its consequences were discussed as well.

For given function  $\mathbf{x} \in \mathbb{R}^n \mapsto \varphi(\mathbf{x}) \geq 0$ , and an random vector (RV)  $\mathbf{X} \in \mathbb{R}^n$ , with a joint probability density function (PDF)  $f$ , the weighted entropy (WE) of  $\mathbf{X}$  (or  $f$ ) with weight function (WF)  $\varphi$  is defined by

$$h_{\varphi}^w(\mathbf{X}) = h_{\varphi}^w(f) = - \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f(\mathbf{x}) \log f(\mathbf{x}) d\mathbf{x} = -\mathbb{E}_{\mathbf{X}}(\varphi \log f) \quad (1.1)$$

whenever the integral  $\int_{\mathbb{R}^n} \varphi(\mathbf{x}) f(\mathbf{x}) (1 \vee |\log f(\mathbf{x})|) d\mathbf{x} < \infty$ . (A standard agreement  $0 = 0 \cdot \log 0 = 0 \cdot \log \infty$  is adopted throughout the paper). Furthermore, for two functions,  $\mathbf{x} \in \mathbb{R}^n \mapsto f(\mathbf{x}) \geq 0$  and  $\mathbf{x} \in \mathbb{R}^n \mapsto g(\mathbf{x}) \geq 0$ , the relative WE of  $g$  relative to  $f$  with WF  $\varphi$  is defined by

$$D_{\varphi}^w(f||g) = \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f(\mathbf{x}) \log \frac{f(\mathbf{x})}{g(\mathbf{x})} d\mathbf{x}. \quad (1.2)$$

When  $\varphi \equiv 1$  the relative WE yields the Kullback-leibler divergence. Searching the weighted determinant inequalities as direct extension for the standard forms to non-constant weight

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function we refer the reader once more to [2, 3, 16]. However, as consequences of those generalizations a number of new bounds in terms of determinants of positive definite matrices, by applying Gaussian WEs, has been given. As one step further, the author proposed another general form of the WE by introducing the *weighted Renyi entropy*, where in spite of standard case [13], in particular  $p \rightarrow 1$  literally does not intend to the WE but a proportion of it, see [15].

**Definition 1.1** *The  $p$ -th weighted Renyi entropy (WRE) of a RV  $\mathbf{X}$  with PDF  $f$  in  $\mathbb{R}^n$ , given WF  $\varphi$  and for  $p > 0, p \neq 1$ , is defined by*

$$h_{\varphi,p}^w(\mathbf{X}) := h_{\varphi,p}^w(f) = \frac{1}{1-p} \log \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f^p(\mathbf{x}) d\mathbf{x}. \quad (1.3)$$

Observe that if  $\varphi \equiv 1$ , the WRE,  $h_{\varphi,p}^w(f)$ , becomes the known Renyi entropy, denoted by  $h_p(f)$ , cf. [13]. Since  $\varphi \geq 0$ , it can be checked that like Renyi entropy, for  $0 < p < 1$  the WRE is a concave function whereas when  $p > 1$  we can not make a similar statement.

Observe that

$$\lim_{p \rightarrow 1} h_{\varphi,p}^w(f) = h_{\varphi,1}^w(f) = \frac{h_{\varphi}^w(f)}{\mathbb{E}_f[\varphi]}. \quad (1.4)$$

On the other words as  $p \rightarrow 1$  the WRE does not intend to the weighted entropy (WE) precisely, see [5, 11, 19]. Note that both  $h_{\varphi,p}^w(f)$  is a continuous function in  $p$ .

Next, extending the standard notions, the **relative**  $p$ -th weighted Renyi power (WRP) of  $f$  and  $g$  was proposed: for  $p > 0, p \neq 1$  and given WF  $\mathbf{x} \in \mathbb{R}^n \mapsto \varphi(\mathbf{x}) \geq 0$

$$N_{\varphi,p}^w(f, g) = \frac{\left( \int_{\mathbb{R}^n} \varphi g^{p-1} f d\mathbf{x} \right)^{1/(1-p)} \left( \int_{\mathbb{R}^n} \varphi g^p d\mathbf{x} \right)^{1/p}}{\left( \int_{\mathbb{R}^n} \varphi f^p d\mathbf{x} \right)^{1/p(1-p)}}. \quad (1.5)$$

And more generally, for given two functions  $f$  and  $g$  we employed the relative  $p$ -th WRP in order to define the relative  $p$ -th weighted Renyi entropy (WRE) of  $f$  and  $g$  by

$$D_{\varphi,p}^w(f \| g) = \log N_{\varphi,p}^w(f, g). \quad (1.6)$$

Hence one can write

$$D_{\varphi,p}^w(f \| g) = \frac{1}{1-p} \log \left( \int_{\mathbb{R}^n} \varphi g^{p-1} f d\mathbf{x} \right) + \frac{1-p}{p} h_{\varphi,p}^w(g) - \frac{1}{p} h_{\varphi,p}^w(f). \quad (1.7)$$

Particularly Eq. (1.7) yields

$$D_{\varphi,1}^w(f, g) = \lim_{p \rightarrow 1} D_{\varphi,p}^w(f, g) = \frac{D_{\varphi}^w(f \| g)}{\mathbb{E}_f[\varphi]}. \quad (1.8)$$

The reflection of the importance of maximum entropy problems, following the terminology used in [4, 9], features the present paper. We intend to study the Renyi entropy maximizing distributions in weighted case for understanding the advantages and limitations of such extended version of entropy maximizing method. In addition the results presented in the current paper obtains parallels to some of the properties involving determinants of matrices by using the weighted Renyi entropy maximizer. Thus this work is organized

first by reviewing the multivariate Student-t and Student-r distributions. Furthermore, we show that under some constraints in terms of the WF, these distributions maximize the WRE for cases  $p < 1$  and  $p > 1$  distinctly. As a primary consequence, we consider the so-called weighted Hadamard inequality, [16] and extend the bound by taking into account the Pearson II and VII PDFs based on the chain rule for the WRE.

Let us begin with a transparent result as a direct application of Hölder inequality below. In fact, this is  $n$ -dimensional version of Theorem 1.1 in [21], hence the proof is omitted.

**Lemma 1.1** *For  $p > 0$  and PDFs  $f, g$  and given WF  $\varphi$ , assume if  $p = 1$ ,  $\mathbb{E}_f[\varphi] \geq \mathbb{E}_g[\varphi]$  holds. The relative  $p$ -th WRE  $D_{\varphi,p}^w(f||g) \geq 0$ . Equality occurs when  $f \equiv g$  almost everywhere.*

Reviewing the Renyi maximizing densities, Student-t (Pearson type VII) and Student-r (Pearson type II) distributions, we use the same notation as in [9],  $g_{p,\mathbf{C}}$ , and establish the following definition. For some of their properties the reader may refer to [4]. Note that we will set  $x_+ = \max\{x, 0\}$ .

**Definition 1.2** *Define the  $n$ -dimensional PDF  $g_{p,\mathbf{C}}$  as*

$$g_{p,\mathbf{C}}(\mathbf{x}) = \begin{cases} A_p \left(1 + (1-p)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}\right)_+^{\frac{1}{p-1}}, & p > n/(n+2), \quad p \neq 1, \\ ((2\pi)^n \det \mathbf{C})^{-1/2} \exp\left\{-\frac{1}{2} \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}\right\}, & p = 1. \end{cases} \quad (1.9)$$

where

$$\beta = \frac{1}{2p - n(1-p)}, \quad \text{and} \quad (1.10)$$

$$A_p = \begin{cases} \left( \Gamma\left(\frac{1}{1-p}\right) (\beta(1-p))^{n/2} \right) / \left( \Gamma\left(\frac{1}{1-p} - \frac{n}{2}\right) \pi^{n/2} (\det \mathbf{C})^{1/2} \right), & \frac{n}{n+2} < p < 1, \\ \left( \Gamma\left(\frac{p}{p-1} + \frac{n}{2}\right) (\beta(p-1))^{n/2} \right) / \left( \Gamma\left(\frac{p}{p-1}\right) \pi^{n/2} (\det \mathbf{C})^{1/2} \right), & p > 1. \end{cases}$$

Here  $\Gamma$  stands the Gamma function. For brevity we will use  $\mathbb{S}_{p,\mathbf{C}}$  for the support of PDF  $g_{p,\mathbf{C}}$ , hence if  $p < 1$ ,  $\mathbb{S}_{p,\mathbf{C}} = \mathbb{R}^n$  and for  $p > 1$ ,  $\mathbb{S}_{p,\mathbf{C}} = \{\mathbf{x}, \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \leq 2p/(p-1) + n\}$ .

We briefly study the Pearson's type II and VII multivariate distributions by referring to [25], which we will recall them throughout the paper. From now on, because of homogeneity, we shall use only Pearson's type II and VII names for these kind of PDFs.

Let  $\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x}^T \mathbf{x} \leq 1\}$ , with  $\mathbb{R}^n$  being the  $n$ -dimensional Euclidian space. Then the Pearson's type II and VII with parameter  $\mu$ , denoted by  $f_{II}(\mathbf{x}; \mu)$ ,  $f_{VII}(\mathbf{x}; \mu)$ , are defined as follows respectively:

$$f_{II}(\mathbf{x}; \mu) = \frac{\Gamma(n/2 + \mu + 1)}{\pi^{n/2} \Gamma(\mu + 1)} (1 - \mathbf{x}^T \mathbf{x})^\mu, \quad \mathbf{x} \in \mathbb{S}, \quad \mu > -1, \quad (1.11)$$

$$f_{VII}(\mathbf{x}; \mu) = \frac{\Gamma(\mu)}{\pi^{n/2} \Gamma(\mu - n/2)} (1 + \mathbf{x}^T \mathbf{x})^{-\mu}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \mu > n/2.$$

Furthermore, for  $n/(n+2) < p < 1$ ,  $q > (1-p)n/2$  set

$$\varpi_n^*(p, q) = \frac{\Gamma^q(1/(1-p)) (\beta(1-p))^{n(q-1)/2} \Gamma(q/(1-p) - n/2)}{\Gamma^q(1/(1-p) - n/2) \pi^{n(q-1)/2} \Gamma(q/(1-p))}, \quad (1.12)$$

and for  $p > 1$ ,  $q > 0$  denote

$$\varpi_n(p, q) = \frac{\Gamma^q(p/(1-p) + n/2) (\beta(p-1))^{n(q-1)/2} \Gamma(q/(p-1) + 1)}{\Gamma^q(p/(p-1)) \pi^{n(q-1)/2} \Gamma(n/2 + q/(p-1) + 1)}. \quad (1.13)$$

Here  $\beta$  is as before (1.10). Also note that we will use  $\varpi_n^*(p)$ ,  $\varpi_n(p)$  when in (1.12) and (1.13)  $p = q$ . Accordingly, let  $\mathbf{Y} = (Y_1, \dots, Y_n)$ ,  $\mathbf{Y}^* = (Y_1^*, \dots, Y_n^*)$  be RVs with PDFs Pearson's type II and VII, i.e.  $\mathbf{Y} \sim f_{II}(\cdot; q/(p-1))$ ,  $\mathbf{Y}^* \sim f_{VII}(\cdot; q/(1-p))$ . We then introduce for  $n/(n+1) < p < 1$ ,  $q > (1-p)n/2$

$$\begin{aligned} \rho^* &:= \rho_{\varphi,p}^*(\mathbf{Y}^*) = \varphi\left(\left\{2(p/(1-p) - n/2)\right\}^{1/2} \mathbf{C}^{1/2} \mathbf{Y}^*\right), \quad \alpha_{\varphi,p}^*(\mathbf{C}) = \mathbb{E}_{f_{VII}} [\rho_{\varphi,p}^*(\mathbf{Y}^*)], \\ \text{and for } p > 1, q > 0 & \\ \rho &:= \rho_{\varphi,p}(\mathbf{Y}) = \varphi\left(\left\{2(p/(p-1) + n/2)\right\}^{1/2} \mathbf{C}^{1/2} \mathbf{Y}\right), \quad \alpha_{\varphi,p}(\mathbf{C}) = \mathbb{E}_{f_{II}} [\rho_{\varphi,p}(\mathbf{Y})]. \end{aligned} \quad (1.14)$$

In addition, it is worthwhile to mention that in particular choice  $\varphi(\mathbf{x}) = \prod_{i=1}^n \varphi_i(x_i)$  where  $x_i \in \mathbb{R} \mapsto \varphi_i(x_i) \geq 0$ , the expression (1.14) takes the forms

$$\begin{aligned} \alpha_{\varphi,p}^*(\mathbf{C}) &= \mathbb{E}_{f_{VII}} \left[ \prod_{i=1}^n \varphi_i \left( \left\{2(p/(1-p) - n/2)\right\}^{1/2} \sum_{j=1}^n Y_j^* C_{ij}^{1/2} \right) \right], \quad n/(n+1) < p < 1, \\ \alpha_{\varphi,p}(\mathbf{C}) &= \mathbb{E}_{f_{II}} \left[ \prod_{i=1}^n \varphi_i \left( \left\{2(p/(p-1) + n/2)\right\}^{1/2} \sum_{j=1}^n Y_j C_{ij}^{1/2} \right) \right], \quad p > 1. \end{aligned} \quad (1.15)$$

Going back to the Definition 1.2, we continue here the section by establishing the Weighted Renyi entropy for Renyi entropy maximizer and given WF  $\varphi$ . Regarding the Pearson distributions suppose that  $\mathbf{X} \sim g_{p,\mathbf{C}}$ , therefore for  $n/(n+2) < p < 1$ ,  $q > (1-p)n/2$  one gets

$$h_{\varphi,q}^w(g_{p,\mathbf{C}}) = \frac{1}{1-q} \log \varpi_n^*(p, q) + \frac{1}{2} \log \det \mathbf{C} + \frac{1}{1-q} \log \alpha_{\varphi,p}^*(\mathbf{C}). \quad (1.16)$$

Moreover, if  $p > 1$ ,  $q > 0$  one obtains

$$h_{\varphi,q}^w(g_{p,\mathbf{C}}) = \frac{1}{1-q} \log \varpi_n(p, q) + \frac{1}{2} \log \det \mathbf{C} + \frac{1}{1-q} \log \alpha_{\varphi,p}(\mathbf{C}). \quad (1.17)$$

## 2 Maximum weighted Renyi entropy

As we said in the introduction, one of our goal in this work is to analyze the maximum weighted Renyi entropy. Precisely, following standard arguments, we first extend the result of Proposition 1.3, [9], to exponents of the maximum WRE for RV  $\mathbf{X}$ .

**Theorem 2.1** *For given  $p > n/(n+2)$ , consider RV  $X$  with PDF  $f$ , mean  $\mathbf{0}$  and positive definite symmetric covariance matrix  $\mathbf{C}$ . Let  $\mathbf{x} \in \mathbb{R}^n \mapsto \varphi(\mathbf{x})$  be a given positive WF. Define matrices  $n \times n$ ,  $\Psi^f = (\psi_{ij}^f)$  and  $\Psi^g = (\psi_{ij}^g)$  where*

$$\psi_{ij}^f = \int_{\mathbb{R}^n} x_i x_j \varphi(\mathbf{x}) f(\mathbf{x}) \, d\mathbf{x}, \quad \psi_{ij}^g = \int_{\mathbb{R}^n} x_i x_j \varphi(\mathbf{x}) g_{p,\mathbf{C}}(\mathbf{x}) \, d\mathbf{x}.$$

Assume that for  $0 < p < 1$  ( $p > 1$ )

$$\int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) \left[ f(\mathbf{x}) - g_{p,\mathbf{C}}(\mathbf{x}) \right] d\mathbf{x} + (1-p)\beta \operatorname{tr} \left[ \mathbf{C}^{-1} (\Psi^f - \Psi^g) \right] \leq (\geq) 0, \quad (2.1)$$

is fulfilled or consider the WF  $\varphi$  obeys

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \left[ f(x) - g_{1,\mathbf{C}}(\mathbf{x}) \right] d\mathbf{x} &\geq 0, \quad \text{and} \\ \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \log g_{1,\mathbf{C}}(\mathbf{x}) \left[ f(\mathbf{x}) - g_{1,\mathbf{C}}(\mathbf{x}) \right] d\mathbf{x} &\geq 0. \end{aligned} \quad (2.2)$$

Then under constrain (2.1)

$$h_{\varphi,p}^w(f) \leq h_{\varphi,p}^w(g_{p,\mathbf{C}}), \quad p \neq 1, \quad (2.3)$$

holds and under (2.2) one has

$$h_{\varphi}^w(f) \leq h_{\varphi}^w(g_{1,\mathbf{C}}), \quad (2.4)$$

with equality iff  $f \equiv g_{p,\mathbf{C}}$  almost everywhere. In fact the case  $p = 1$  literally illustrates the corresponding result in Example 3.2 cf. [16].

**Proof:** Using the Definition 1.2, for  $0 < p < 1$  ( $p > 1$ ) one can write

$$\begin{aligned} \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) g_{p,\mathbf{C}}^{p-1}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} &= A_p^{p-1} \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) \left( 1 + (1-p)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \right) f(\mathbf{x}) d\mathbf{x} \\ &\leq (\geq) A_p^{p-1} \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) \left( 1 + (1-p)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \right) g_{p,\mathbf{C}}(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) g_{p,\mathbf{C}}^p(\mathbf{x}) d\mathbf{x}. \end{aligned} \quad (2.5)$$

In expression (2.5), the inequality is emerged from (2.1), and in case  $p = 1$  it becomes

$$\int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) f(\mathbf{x}) \log g_{1,\mathbf{C}}(\mathbf{x}) d\mathbf{x} \geq \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) g_{1,\mathbf{C}}(\mathbf{x}) \log g_{1,\mathbf{C}}(\mathbf{x}) d\mathbf{x}. \quad (2.6)$$

Now recall Lemma 1.1. Therefore owing to (1.7) one yields

$$\begin{aligned} 0 \leq D_{\varphi,p}^w(f \| g_{p,\mathbf{C}}) &= \frac{1}{1-p} \log \left( \int_{\mathbb{R}^n} \varphi g_{p,\mathbf{C}}^{p-1} f d\mathbf{x} \right) + \frac{1-p}{p} h_{\varphi,p}^w(g_{p,\mathbf{C}}) - \frac{1}{p} h_{\varphi,p}^w(f) \\ &\leq \frac{1}{p} \left( h_{\varphi,p}^w(g_{p,\mathbf{C}}) - h_{\varphi,p}^w(f) \right). \end{aligned} \quad (2.7)$$

Likewise, the proof in case  $p = 1$  follows by repeating verbatim in the n-dimensional setting in Example 3.2 from [16].  $\blacksquare$

**Remark 2.1** Considering arguments in [4], for given WF  $\varphi$ , we introduce the non-symmetric directed divergence measure (see [6], [1]) for weighted case by

$$D_{\varphi,p}^w(f \| g) = \operatorname{sign}(p-1) \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) \left( \frac{f^p(\mathbf{x})}{p} + \frac{p-1}{p} g^p(\mathbf{x}) - f(\mathbf{x}) g^{p-1}(\mathbf{x}) \right) d\mathbf{x}. \quad (2.8)$$

Going back to (2.5), with the same strategy as Theorem 2.1, one deduces that under condition (2.1),  $D_{\varphi,p}^w(f \| g) \geq 0$ . This assertion implies (2.3), which can be considered as alternative proof for Theorem 2.1.

In sequel, in terms of a multivariate RV  $\mathbf{X}$ , let us introduce

$$\begin{aligned}\eta_{\varphi,p}^*(\mu) &= \mathbb{E}_{f_{VII}} \left[ \varphi \log (1 + \mathbf{X}^T \mathbf{X}) \right], \quad n/(n+2) < p < 1, \\ \eta_{\varphi,p}(\mu) &= \mathbb{E}_{f_{II}} \left[ \varphi \log (1 - \mathbf{X}^T \mathbf{X}) \right], \quad p > 1.\end{aligned}\tag{2.9}$$

where  $\mu = p/(p-1)$ . Using results obtained in [24], next we shall follow the analogue methodology and apply the generalized spherical coordinate transformation. Therefore we are able to establish the explicit quantities for various forms of the WF  $\varphi$ . Let us begin with  $\varphi(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$ , so we compute

$$\begin{aligned}\eta_{\varphi,p}^*(\mu) &= \frac{\Gamma(\mu)}{\pi^{n/2} \Gamma(\mu - n/2)} \int_{\mathbb{R}^n} \mathbf{x}^T \mathbf{x} (1 + \mathbf{x}^T \mathbf{x})^{-\mu} \log (1 + \mathbf{x}^T \mathbf{x}) d\mathbf{x} \\ &= \frac{2\Gamma(\mu)}{\Gamma(n/2) \Gamma(\mu - n/2)} \int_0^\infty r^{n+1} (1 + r^2)^{-\mu} \log(1 + r^2) dr \\ &= \frac{n}{2\mu - n - 2} \left\{ \Psi(\mu) - \Psi(\mu - n/2 - 1) \right\}, \quad \mu > n/2 + 1.\end{aligned}\tag{2.10}$$

Here  $\Psi(t) = \frac{d}{dt} \log \Gamma(t)$ . The last line in (2.10) is derived by differentiating the Beta function defined by the following integral with respect to  $\beta$ :

$$B(\alpha, \beta) = \int_0^\infty t^{\alpha-1} (1+t)^{-\alpha-\beta} dt, \quad \alpha > 0, \beta > 0.$$

Similarly one yields

$$\eta_{\varphi,p}(\mu) = \frac{n}{2\mu + n + 2} \left\{ \Psi(\mu + 1) - \Psi(\mu + n/2 + 2) \right\}, \quad \mu > -1.\tag{2.11}$$

Further, consider  $\varphi(\mathbf{x}) = \log \mathbf{x}^T \mathbf{x}$ , then for  $\mu > -1$ , straightforwardly we can write

$$\eta_{\varphi,p}(\mu) = \left( \Psi(\mu + 1) - \Psi(\mu + n/2 + 1) \right) \left( \Psi(n/2) - \Psi(\mu + n/2 + 1) \right) - \Psi'(\mu + n/2 + 1).$$

Here  $\Psi'$  stands the derivative function of  $\Psi$ .

**Theorem 2.2** Suppose  $f$  is a PDF and  $\mathbf{x} \in \mathbb{R}^n \mapsto \varphi(\mathbf{x}) \geq 0$ . Then  $g_{p,\mathbf{C}}$  is the unique maximizer of the WE  $h_\varphi^w(f)$  under constraints

$$\begin{aligned}\mathbb{E}_{g_{p,\mathbf{C}}}[\varphi] &\leq \mathbb{E}_f[\varphi], \quad \text{and} \\ \int_{\mathbb{S}_{p,\mathbf{C}}} \varphi(\mathbf{x}) f(\mathbf{x}) \log \left( 1 + (1-p)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \right) d\mathbf{x} \\ &\leq \eta_{\rho^*,p}^* \left( \frac{1}{1-p} \right) + (1-p) \log A_p \left( \mathbb{E}_f[\varphi] - \mathbb{E}_{g_{p,\mathbf{C}}}[\varphi] \right), \quad \text{if } p < 1, \\ &\geq \eta_{\rho,p} \left( \frac{1}{p-1} \right) + (1-p) \log A_p \left( \mathbb{E}_f[\varphi] - \mathbb{E}_{g_{p,\mathbf{C}}}[\varphi] \right), \quad \text{if } p > 1.\end{aligned}\tag{2.12}$$

with equality  $f \equiv g_{p,\mathbf{C}}$ . Here  $\rho^*, \rho$  form as (1.14) and  $\eta^*, \eta$  stands as before, (2.9).

The next step is to recall Theorem 6. from [4], omitting the proof.

**Theorem 2.3** Let  $\mathbf{X}, \mathbf{Y}$  be two independent RVs with covariance matrices  $\mathbf{C}_{\mathbf{X}} = \mathbf{C}_{\mathbf{Y}} = \mathbf{I}_n$  and odd degrees of freedom  $p_{\mathbf{x}}, p_{\mathbf{y}}$  and PDFs  $g_{p_{\mathbf{x}}, \mathbf{I}}, g_{p_{\mathbf{y}}, \mathbf{I}}$  respectively, where

$$g_{p_{\mathbf{x}}, \mathbf{C}}(\mathbf{x}) = (p_{\mathbf{x}} - 2)^{-n/2} g_{p, \mathbf{C}}((p_{\mathbf{x}} - 2)^{-1/2} \mathbf{x}), \quad p = \frac{p_{\mathbf{x}} + n - 2}{p_{\mathbf{x}} + n}.$$

Then for  $0 \leq \lambda \leq 1$ , the distribution of  $\mathbf{Z} = \lambda \mathbf{X} + (1 - \lambda) \mathbf{Y}$  is

$$g_{\mathbf{Z}}(\mathbf{z}) = \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k g_{p_{2k+1}, \mathbf{I}}(\mathbf{z}). \quad (2.13)$$

where  $p_{\mathbf{z}} \leq \frac{p_{\mathbf{x}} + p_{\mathbf{y}}}{2} - 1$ . Note that throughout the note we shall use  $g_p$  instead of  $g_{p, \mathbf{I}}$  as well.

**Theorem 2.4** Suppose that  $\mathbf{X}, \mathbf{Y}$  are two independent RVs with PDFs  $g_{p_{\mathbf{x}}}, g_{p_{\mathbf{y}}}$ , such that  $\mathbf{C}_{\mathbf{X}} = \mathbf{C}_{\mathbf{Y}} = \mathbf{I}_n$  and  $p_{\mathbf{x}}, p_{\mathbf{y}}$  are odd freedom degrees. Assume RV  $\mathbf{Z} = \frac{\mathbf{X} + \mathbf{Y}}{2}$ . Set

$$\Delta_n(p) = \Psi\left(\frac{p+n}{2}\right) - \Psi\left(\frac{p}{2}\right). \quad (2.14)$$

Then regarding the weighted Kullback-Leibler divergence the distribution  $g_{p^*}$  with freedom degree  $p^*$  which obeys

$$\Delta_n(p^*) = \mathbb{E}_{\mathbf{W}}\left(\mathbb{E}_{g_{p_{2k+1}}}[\varphi \log(1 + \mathbf{Z}^T \mathbf{Z})]\right) / \mathbb{E}_{\mathbf{W}}\left(\mathbb{E}_{g_{p_{2k+1}}}[\varphi]\right), \quad (2.15)$$

or equivalently

$$\mathbb{E}_{g_{p^*}}[\log(1 + \mathbf{X}^T \mathbf{X})] \mathbb{E}_{g_{\mathbf{Z}}}[\varphi] = \mathbb{E}_{g_{\mathbf{Z}}}[\varphi(\mathbf{X}) \log(1 + \mathbf{X}^T \mathbf{X})], \quad (2.16)$$

is the closest to the distribution  $\mathbf{Z}$ ,  $g_{\mathbf{Z}}$ . Here the RV  $\mathbf{W}$  is distributed as

$$P(\mathbf{W} = 2k + 1) = \alpha_k. \quad (2.17)$$

**Proof:** Taking into account (1.2), observe that for  $g_{p^*}$  one can write

$$D_{\phi}^{\mathbf{W}}(g_{\mathbf{Z}} \| g_{p^*}) = -h_{\varphi}^{\mathbf{W}}(g_{\mathbf{Z}}) - \int_{\mathbb{R}^n} \varphi(\mathbf{z}) g_{\mathbf{Z}}(\mathbf{z}) \log g_{p^*}(\mathbf{z}) d\mathbf{z}. \quad (2.18)$$

In order to explore the optimal value for  $p^*$  we minimizes (2.18). This is equivalent to find the maximizer of the above integral. For this reason we focus on the  $\int_{\mathbb{R}^n} \varphi(\mathbf{z}) g_{\mathbf{Z}}(\mathbf{z}) \log g_{p^*}(\mathbf{z}) d\mathbf{z}$ .

Thus one derives

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(\mathbf{z}) g_{\mathbf{Z}}(\mathbf{z}) \log g_{p^*}(\mathbf{z}) d\mathbf{z} &= \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k \int_{\mathbb{R}^n} \varphi(\mathbf{z}) g_{p_{2k+1}}(\mathbf{z}) \log g_{p^*}(\mathbf{z}) d\mathbf{z} \\ &= \left(\log A'_{p^*}\right) \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k \int_{\mathbb{R}^n} \varphi(\mathbf{z}) g_{p_{2k+1}}(\mathbf{z}) d\mathbf{z} \\ &+ \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k \int_{\mathbb{R}^n} \varphi(\mathbf{z}) g_{p_{2k+1}}(\mathbf{z}) \log \left(1 + \mathbf{z}^T \mathbf{z}\right)^{-(p^*+n)/2} d\mathbf{z}. \end{aligned} \quad (2.19)$$

Substituting

$$A'_{p^*} = \Gamma\left(\frac{p^* + n}{2}\right) / \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{p^*}{2}\right)$$

in expression (2.19), the LHS turns into

$$\sum_{k=0}^{p_{\mathbf{z}}} \alpha_k \mathbb{E}_{g_{p_{2k+1}}}[\varphi] \log \frac{\Gamma\left(\frac{p^*+n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p^*}{2}\right)} - \frac{p^*+n}{2} \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k \mathbb{E}_{g_{p_{2k+1}}} \left[ \varphi \log(1 + \mathbf{Z}^T \mathbf{Z}) \right]. \quad (2.20)$$

Taking the derivative of (2.20) with respect to  $p^*$  obtains

$$\frac{\Delta_n(p^*)}{2} \mathbb{E}_{\mathbf{W}} \left( \mathbb{E}_{g_{p_{2k+1}}}[\varphi] \right) - \frac{1}{2} \mathbb{E}_{\mathbf{W}} \left( \mathbb{E}_{g_{p_{2k+1}}} \left[ \varphi \log(1 + \mathbf{Z}^T \mathbf{Z}) \right] \right), \quad (2.21)$$

where  $\Delta_n(p^*)$  reads (2.14) and  $\mathbf{W}$  is denoted for a RV with distribution according to (2.17). Equating (2.21) to zero the desired result is achieved. Now it only remains to check the second derivative of (2.21). We know that the derivative function of  $\Psi$  is non-increasing, thus

$$\frac{\partial}{\partial p} \Delta_n(p) = \frac{1}{2} \Psi' \left( \frac{p+n}{2} \right) - \frac{1}{2} \Psi' \left( \frac{p}{2} \right) \leq 0.$$

In addition, following the arguments in [24], [4], one has

$$\Delta_n(p^*) = \mathbb{E}_{g_{p^*}} \left[ \log (1 + \mathbf{X}^T \mathbf{X}) \right].$$

This together with

$$\begin{aligned} \mathbb{E}_{\mathbf{W}} \left( \mathbb{E}_{g_{p_{2k+1}}} \left[ \varphi \log(1 + \mathbf{Z}^T \mathbf{Z}) \right] \right) &= \int_{\mathbb{R}^n} \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k g_{p_{2k+1}}(\mathbf{z}) \varphi(\mathbf{z}) \log (1 + \mathbf{z}^T \mathbf{z}) d\mathbf{z} \\ &= \mathbb{E}_{g_{\mathbf{Z}}} \left[ \varphi(\mathbf{Z}) \log (1 + \mathbf{Z}^T \mathbf{Z}) \right], \end{aligned}$$

and

$$\mathbb{E}_{\mathbf{W}} \left( \mathbb{E}_{g_{p_{2k+1}}}[\varphi] \right) = \int_{\mathbb{R}^n} \sum_{k=0}^{p_{\mathbf{z}}} \alpha_k g_{p_{2k+1}}(\mathbf{z}) \varphi(\mathbf{z}) d\mathbf{z} = \mathbb{E}_{g_{\mathbf{Z}}} [\varphi(\mathbf{Z})].$$

leads directly to (2.16).  $\blacksquare$

In the remaining arguments of this section, we shall address the reader to the following lemma as a technical low bound for the WRE.

**Lemma 2.1** *Assume the sequence of PDFs  $f_1, \dots, f_m$  on  $\mathbb{R}^n$ . Consider constants  $s_1, \dots, s_m$  such that  $\sum_{i=1}^m s_i = 1$ . For the mixture PDF  $f$ ,*

$$f(\mathbf{x}) = \sum_{i=1}^m s_i f_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

and given WF  $\varphi$ , we have

$$h_{\varphi,p}^{\mathbf{w}}(f) \geq \min_{1 \leq i \leq m} h_{\varphi,p}^{\mathbf{w}}(f_i). \quad (2.22)$$



**Proof:** We begin with case  $0 < p < 1$  which holds because of the concavity property for the WRE. Next suppose that  $p > 1$ , then we can write

$$\begin{aligned} \log \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f^p(\mathbf{x}) d\mathbf{x} &= \log \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \left( \sum_i s_i f_i(\mathbf{x}) \right)^p d\mathbf{x} \\ &\leq \log \sum_i s_i \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f_i^p(\mathbf{x}) d\mathbf{x} \\ &\leq \log \max_{1 \leq i \leq m} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f_i^p(\mathbf{x}) d\mathbf{x} \\ &= \max_{1 \leq i \leq m} \log \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f_i^p(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

The proof of theorem is completed by observing that  $\frac{1}{1-p}$  is negative.  $\blacksquare$

### 3 Extended Hadamard inequality and its consequences

In this section, our aim is to establish a generalized form for Hadamard inequality in terms of WRE for given WF  $\varphi$ . Combining this with maximum WRE distributions leads us to the assertions claimed in Corollary 3.21, 3.2. The following lemma is an immediate application of Lemma 1.1.

**Lemma 3.1** *Let  $\mathbf{X}$  be a RV with PDF  $f$  with components  $X_i; \Omega \mapsto \mathbb{R}, 1 \leq i \leq n$  having marginal PDF  $f_i$ , and joint PDF  $f$ . Given the WFs  $\varphi_i \geq 0$ , such that  $\varphi(\mathbf{x}) = \prod_{i=1}^n \varphi_i(x_i)$  is considered as the WF. Suppose that*

$$\int_{\mathbb{R}^n} \prod_{i=1}^n \varphi_i(x_i) f_i^{p-1}(x_i) \left[ f(\mathbf{x}) - \prod_{i=1}^n f_i(x_i) \right] d\mathbf{x} \leq (\geq) 0, \quad \text{for } 0 < p < 1 \text{ (} p > 1 \text{)}. \quad (3.1)$$

Then

$$h_{\varphi,p}^w(f) \leq \sum_{i=1}^n h_{\varphi_i,p}^w(f_i). \quad (3.2)$$

The equality here holds iff the components  $X_1, \dots, X_n$  are independent.

**Theorem 3.1** (The extended Hadamard inequality, Theorem 3.9 cf. [16]). *Let  $\mathbf{C} = (C_{ij})$  be a positive definite  $n \times n$  matrix and  $g_{p,\mathbf{C}}$  stands the PDF in Definition 1.2. In addition let  $g_{p,C_{ii}}$  be the marginal PDF of the  $i$ -th components, that is as in (1.9) when  $n = 1$ . Then for given functions  $x_i \in \mathbb{R} \mapsto \varphi_i(x_i) \geq 0, 1 \leq i \leq n$  which if  $n/(n+2) < p < 1$  ( $p > 1$ ) obey*

$$\int_{\mathbb{S}_{p,\mathbf{C}}} \prod_{i=1}^n \varphi_i(x_i) g_{p,C_{ii}}^{p-1}(x_i) \left[ g_{p,\mathbf{C}}(\mathbf{x}) - \prod_{i=1}^n g_{p,C_{ii}} \right] d\mathbf{x} \leq (\geq) 0, \quad (3.3)$$

one has,  $n/(n+2) < p < 1$ ,

$$\begin{aligned} &\frac{1-p}{2} \log \prod_i C_{ii} + \sum_i \log \varpi_1^*(p) \alpha_{\varphi_i,p}^*(C_{ii}) \\ &\quad - \frac{1-p}{2} \log \det \mathbf{C} - \log \varpi_n^*(p) \alpha_{\varphi,p}^*(\mathbf{C}) \geq 0. \end{aligned} \quad (3.4)$$

Here  $\varpi_n^*(p), \alpha_{\varphi,p}^*(\mathbf{C})$  are as in (1.12), (1.15) and

$$\alpha_{\varphi_i,p}^*(C_{ii}) = \mathbb{E} \left[ \varphi_i \left( \left\{ C_{ii}((3p-1)/(1-p)) \right\}^{1/2} Y_i^* \right) \right], \quad p \in (1/3, 1). \quad (3.5)$$

where  $Y_i^*$  has the Pearson's type VII univariate distribution with parameter  $\mu = p/(1-p)$ . For case  $p > 1$ , recall (1.13), (1.15) and in (3.4) swap  $\varpi_n(p)$ ,  $\alpha_{\varphi,p}(\mathbf{C})$ ,  $\alpha_{\varphi,p}(C_{ii})$  with  $\varpi_n^*(p)$ ,  $\alpha_{\varphi,p}^*(\mathbf{C})$ ,  $\alpha_{\varphi,p}^*(C_{ii})$ . Note that  $\alpha_{\varphi,p}(C_{ii})$  is defined in similar manner as (3.5) by replacing random variable  $Y_i$  in  $Y_i^*$  where  $Y_i$  has the Pearson's type II univariate distribution with parameter  $p/(p-1)$ . The equality in (3.4) occurs iff  $\mathbf{C}$  is diagonal.

**Proof:** We give the proof for part  $n/(n+2) < p < 1$ , while the proof for the case  $p > 1$  follows in a similar manner. Assume that RV  $\mathbf{X}$  has PDF  $g_{p,\mathbf{C}}$ . By virtue of (3.2), (1.16) one yields

$$\begin{aligned} & \frac{1}{1-p} \log \varpi_n^*(p) + \frac{1}{2} \log \det \mathbf{C} + \frac{1}{1-p} \log \alpha_{\varphi,p}^*(\mathbf{C}) \\ & \leq \sum_{i=1}^n \left( \frac{1}{1-p} \log \varpi_1^*(p) + \frac{1}{2} \log C_{ii} + \frac{1}{1-p} \log \alpha_{\varphi,p}^*(C_{ii}) \right). \end{aligned}$$

Here quantity  $\alpha_{\varphi,p}^*(C_{ii})$  is introduced by (3.5). The assertion (3.4) then follows. Note that the case equality is covered by the equality in Lemma 3.1. ■

Now, we recall the following definition from [14], which is essentially the integral representation of the modified Bessel function of the third and first kinds. ( see [23], p. 182).

**Definition 3.1** (a) *The integral representation of the modified Bessel function of the third kind, denoted as  $K_\lambda(z)$ , is defined by*

$$K_\lambda(z) = \frac{1}{2} \int_0^\infty x^{\lambda-1} \exp \left\{ -\frac{1}{2} z \left( x + \frac{1}{x} \right) \right\} dx, \quad z > 0. \quad (3.6)$$

Note that  $K_\lambda(z) = K_{-\lambda}(z)$ ,  $z > 0$ ,  $\lambda \in \mathbb{R}$  and

$$K_\lambda(z) \cong \Gamma(\lambda) 2^{\lambda-1} z^{-\lambda}, \quad \text{as } z \rightarrow 0^+, \lambda > 0.$$

(b) *The Bessel function of the first kind, written as  $J_\gamma(z)$ , is given in form*

$$J_\gamma(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta. \quad (3.7)$$

The techniques developed by the WRE in Theorem 3.1 so far allow us to establish Corollary 3.21 below rendering a general form of Hadamard inequality. To this end, we firstly introduce more notations:

$$\epsilon_n = 2 \left( \frac{p}{1-p} - \frac{n}{2} \right) \quad \text{and} \quad \chi_n^*(p) = \frac{\varpi_n^*(p)}{\Gamma(\epsilon_n/2) 2^{\epsilon_n/2-1}}, \quad n/(n+2) < p < 1. \quad (3.8)$$

In (3.8), by setting  $n = 1$  we get  $\epsilon_1$  and  $\chi_1^*(p)$ . Also set

$$\xi_n = 2 \left( \frac{p}{p-1} + \frac{n}{2} \right) \quad \text{and} \quad \chi_n(p) = 2^{\xi_n/2} \Gamma(\xi_n/2 + 1) \varpi_n(p), \quad p > 1. \quad (3.9)$$

Observe that  $\xi_1$  and  $\chi_1(p)$  are obtained if in (3.9) we consider  $n = 1$ .

**Corollary 3.1** *Suppose  $\mathbf{C}$  is a positive definite  $n \times n$  matrix. Consider the vector  $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$  is satisfied in*

$$\int_{\mathbb{S}_{p,\mathbf{C}}} e^{i\mathbf{t}\mathbf{x}} \prod_{i=1}^n g_{p,C_{ii}}^{p-1}(x_i) \left[ g_{p,\mathbf{C}}(\mathbf{x}) - \prod_{i=1}^n g_{p,C_{ii}} \right] d\mathbf{x} \leq (\geq) 0, \quad n/(n+2) < p < 1 (p > 1) \quad (3.10)$$

Invoking  $K_\lambda(z)$  in Definition 3.1 gives the bound

$$\begin{aligned} & \frac{1-p}{2} \log \prod_i C_{ii} + \sum_i \log \left( \chi_1^*(p) K_{\epsilon_1/2} \left( \sqrt{\epsilon_1 C_{ii}} |t_i| \right) \left( \sqrt{\epsilon_1 C_{ii}} |t_i| \right)^{\epsilon_1/2} \right) \\ & - \frac{1-p}{2} \log \det \mathbf{C} - \log \left( \chi_n^*(p) K_{\epsilon_n/2} \left( \|\sqrt{\epsilon_n} \mathbf{C}^{1/2} \mathbf{t}\| \right) \left( \|\sqrt{\epsilon_n} \mathbf{C}^{1/2} \mathbf{t}\| \right)^{\epsilon_n/2} \right) \geq 0, \end{aligned} \quad (3.11)$$

when  $n/(n+2) < p < 1$ . One can also for  $p > 1$  deduce

$$\begin{aligned} & \frac{1-p}{2} \log \prod_i C_{ii} + \sum_i \log \left( \chi_1(p) J_{\xi_1/2} \left( \sqrt{\xi_1 C_{ii}} |t_i| \right) \left( \sqrt{\xi_1 C_{ii}} |t_i| \right)^{\xi_1/2} \right) \\ & - \frac{1-p}{2} \log \det \mathbf{C} - \log \left( \chi_n(p) J_{\xi_n/2} \left( \|\sqrt{\xi_n} \mathbf{C}^{1/2} \mathbf{t}\| \right) \left( \|\sqrt{\xi_n} \mathbf{C}^{1/2} \mathbf{t}\| \right)^{\xi_n/2} \right) \geq 0, \end{aligned} \quad (3.12)$$

Here  $J_\cdot$  refers to the Bessel function of the first kind indicated in (3.7). Further particularly, for a positive definite  $2 \times 2$  matrix one has

$$\frac{1}{6} \log \frac{C_{11}C_{22}}{\det \mathbf{C}} + \log \frac{\prod_{i=1}^2 K_{3/2} \left( \sqrt{3C_{ii}} |t_i| \right) \left( \sqrt{3C_{ii}} |t_i| \right)^{3/2}}{K_1 \left( \|\sqrt{2} \mathbf{C}^{1/2} \mathbf{t}\| \right) \left( \|\sqrt{2} \mathbf{C}^{1/2} \mathbf{t}\| \right)} \geq \log \frac{3\pi^{2/3}}{4}. \quad (3.13)$$

**Proof:** The proof for part  $n/(n+2) < p < 1$  is provided. By using  $\varphi_l(x_l) = e^{it_l x_l}$  in (1.15), for  $\mathbf{Y}^* \sim f_{V_{II}}$  and  $\mathbf{t} \in \mathbb{R}^n$ , we have

$$\alpha_{\varphi,p}^*(\mathbf{C}) = \mathbb{E}_{f_{V_{II}}} \left[ \prod_{l=1}^n e^{i\sqrt{\epsilon_n} t_l \sum_j Y_j^* C_{lj}^{1/2}} \right] = \phi_{\mathbf{Y}^*}(\sqrt{\epsilon_n} \mathbf{C}^{1/2} \mathbf{t}),$$

where  $\epsilon_n$  stands as in (3.8) and  $\phi_{\mathbf{Y}^*}$  represents the characteristic function for RV  $\mathbf{Y}^*$ . By virtue of Result 4, cf. [14] one yields

$$\alpha_{\varphi,p}^*(\mathbf{C}) = \frac{K_{\epsilon_n/2} \left( \|\sqrt{\epsilon_n} \mathbf{C}^{1/2} \mathbf{t}\| \right) \left( \|\sqrt{\epsilon_n} \mathbf{C}^{1/2} \mathbf{t}\| \right)^{\epsilon_n/2}}{\Gamma(\epsilon_n/2) 2^{\epsilon_n/2-1}}. \quad (3.14)$$

Similarly, in accordance with (3.5) one can derive

$$\begin{aligned} \alpha_{\varphi_l,p}^*(C_{ll}) &= \mathbb{E} \left[ e^{i\sqrt{\epsilon_1} C_{ll} t_l Y_l^*} \right] = \phi_{Y_l^*} \left( \sqrt{\epsilon_1 C_{ll}} t_l \right) \\ &= \frac{K_{\epsilon_1/2} \left( \sqrt{\epsilon_1 C_{ll}} |t_l| \right) \left( \sqrt{\epsilon_1 C_{ll}} |t_l| \right)^{\epsilon_1/2}}{\Gamma(\epsilon_1/2) 2^{\epsilon_1/2-1}}. \end{aligned} \quad (3.15)$$

Replace (3.14), (3.15) in (3.4). This concludes the proof. The assertion (3.13) is achieved by choosing  $p = \frac{2}{3}$ ,  $n = 2$ . ■

In Lemma 3.2 we extend the results of Lemma 3.1 to exponents of WREs for sub-strings  $(\mathbf{X}_1, \mathbf{X}_2)$  in  $\mathbf{X}$  having  $g_{p,\mathbf{C}}$  PDF. We verify this by owing to Theorem 3 in [4] and Lemma 1.1 straightforwardly, hence the proof is omitted.

**Lemma 3.2** Let  $\mathbf{X}^T = (\mathbf{X}_1^T, \mathbf{X}_2^T)$  be a RV in  $\mathbb{R}^n$ , with PDF  $g_{p,\mathbf{C}}$  and characteristic matrix  $\mathbf{C} = (C_{ij})$ ,  $i, j = 1, 2$ , where  $\dim \mathbf{X}_i = n_i$ ,  $n_1 + n_2 = n$  and  $\dim \mathbf{C}_{ij} = n_i \times n_j$ . Then for given WF  $\varphi(\mathbf{x}) = \prod_{j=1,2} \varphi_j(\mathbf{x}_j)$  which for  $0 < p < 1$  ( $p > 1$ ) is satisfied in below

$$\int_{\mathbb{R}^n} \prod_{j=1,2} \varphi_j(\mathbf{x}_j) g_{p_j, \mathbf{C}_{jj}}^{p-1}(\mathbf{x}_j) \left[ g_{p,\mathbf{C}}(\mathbf{x}) - \prod_{j=1,2} g_{p_j, \mathbf{C}_{jj}}(\mathbf{x}_j) \right] d\mathbf{x} \leq (\geq) 0.$$

where index  $p_j$  is given by

$$\frac{1}{1-p_j} = \frac{1}{1-p} - \frac{n_j}{2}, \text{ and } \mathbf{X}_j \sim g_{p_j, \mathbf{C}_{jj}}.$$

we have

$$h_{\varphi, p}^w(\mathbf{X}) \leq h_{\varphi_1, p}^w(\mathbf{X}_1) + h_{\varphi_2, p}^w(\mathbf{X}_2). \quad (3.16)$$

Next step would be to use Lemma 3.2 and explore an upper bound for determinant of block matrices in terms of the expected value for the WF  $\varphi$ , i.e.  $\alpha_{\varphi, p}$ .

**Theorem 3.2** Consider block matrix  $\mathbf{B} = (\mathbf{B}_{ij})$ ,  $i, j = 1, 2$  with  $\dim \mathbf{B}_{ij} = n'_i \times n_j$ ,  $n = n_1 + n_2$ ,  $n' = n'_1 + n'_2$ . Furthermore let  $\mathbf{C} = (\mathbf{C}_{ij})$ , be positive definite block matrix where  $\dim \mathbf{C}_{ij} = n_i \times n_j$  therefore  $\dim \mathbf{C} = n \times n$ . Assume that  $p'$ ,  $p'_1$ ,  $p'_2$  follow relation

$$\frac{1}{1-p'_i} = \frac{1}{1-p'} - \frac{n'_i}{2}, \text{ whereas } p' > \frac{(1-p'_i) n'_i}{2}, \quad i = 1, 2 \quad (3.17)$$

and take ranges

$$p' \in \left(\frac{n'}{n'+2}, 1\right) \quad \text{and} \quad p'_i \in \left(\frac{n'_i}{n'_i+2}, 1\right), \quad i = 1, 2.$$

Define

$$\zeta(p', p'_1, p'_2) = \frac{\varpi_{n'_1}^*(p'_1, p') \varpi_{n'_2}^*(p'_2, p')}{\varpi_{n'}^*(p')},$$

where  $\varpi^*$ s denote the corresponding quantities in (1.12). Now by recalling  $\alpha^*$  from (1.14) and the given function  $\varphi = \prod_{i=1,2} \varphi_i$ ,  $\varphi_i \geq 0$ , if

$$\int_{\mathbb{R}^{n'}} \prod_{j=1,2} \varphi_j(\mathbf{x}_j) g_{p'_j, \mathbf{C}_{jj}}^{p'-1}(\mathbf{x}_j) \left[ g_{p', \mathbf{BCB}}(\mathbf{x}) - \prod_{j=1,2} g_{p'_j, \mathbf{C}'_j}(\mathbf{x}_j) \right] d\mathbf{x} \leq (\geq) 0. \quad (3.18)$$

holds true then the following inequality is emerged:

$$\left( \frac{\det \mathbf{BCB}^T}{(\det \mathbf{C}'_1) (\det \mathbf{C}'_2)} \right)^{1-p'} \left( \frac{\alpha_{\varphi, p'}^*(\mathbf{BCB}^T)}{\alpha_{\varphi_1, p'_1}^*(\mathbf{C}'_1) \alpha_{\varphi_2, p'_2}^*(\mathbf{C}'_2)} \right)^2 \leq \zeta(p', p'_1, p'_2). \quad (3.19)$$

Here  $\mathbf{C}'_1$ ,  $\mathbf{C}'_2$  with  $\dim \mathbf{C}'_i = n'_i \times n'_i$ ,  $i = 1, 2$  represent the diagonal block matrices  $\mathbf{BCB}^T$ , that is

$$\begin{aligned} \mathbf{C}'_1 &= \sum_{i=1,2} \mathbf{B}_{1i} \mathbf{C}_{ii} \mathbf{B}_{1i}^T + \mathbf{B}_{12} \mathbf{C}_{21} \mathbf{B}_{11}^T + \mathbf{B}_{11} \mathbf{C}_{12} \mathbf{B}_{12}^T, \\ \mathbf{C}'_2 &= \sum_{i=1,2} \mathbf{B}_{2i} \mathbf{C}_{ii} \mathbf{B}_{2i}^T + \mathbf{B}_{22} \mathbf{C}_{21} \mathbf{B}_{21}^T + \mathbf{B}_{21} \mathbf{C}_{12} \mathbf{B}_{22}^T. \end{aligned}$$

**Proof:** Let  $\mathbf{X}_1$  and  $\mathbf{X}_2$  be  $n_1$  and  $n_2$  RVs mutually distributed according to the Renyi entropy maximizing density  $g_{p, \mathbf{C}}$ , where

$$\frac{1}{1-p} = \frac{1}{1-p'} + \frac{n-n'}{2}.$$

Consider RV  $\mathbf{X}^T = (\mathbf{X}_1^T, \mathbf{X}_2^T)$  and deduced RV  $\mathbf{Y} = \mathbf{B} \mathbf{X} = \mathbf{B} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}$ , with  $\mathbf{Y}_i = \mathbf{B}_{i1}\mathbf{X}_1 + \mathbf{B}_{i2}\mathbf{X}_2$ ,  $i = 1, 2$ . Owing to the Theorem 4, [4], the RV  $\mathbf{Y}$  follows  $g_{p', \mathbf{C}'}$  distribution with characteristic matrix  $\mathbf{C}' = \mathbf{B}\mathbf{C}\mathbf{B}^T$ , with  $p'$  given by (3.17). Next by virtue of the Theorem 3 from [4], we can derive that the marginal density of RV  $\mathbf{Y}_i$  is  $g_{p'_i, \mathbf{C}'_{ii}}$  such that  $\dim \mathbf{C}'_{ii} = n'_i \times n'_i$  and  $p'_i$  comes again from (3.17). In this stage recall the Lemma 3.2, therefore one yields

$$h_{\varphi, p'}^w(\mathbf{Y}) \leq h_{\varphi_1, p'}^w(\mathbf{Y}_1) + h_{\varphi_2, p'}^w(\mathbf{Y}_2).$$

Equivalently

$$\begin{aligned} & \frac{1}{1-p'} \log \varpi_{n'}^*(p') + \frac{1}{2} \log \det \mathbf{C}' + \frac{1}{1-p'} \log \alpha_{\varphi, p'}^*(\mathbf{C}') \\ & \leq \frac{1}{1-p'} \log \varpi_{n'_1}^*(p'_1, p') + \frac{1}{2} \log \det \mathbf{C}'_{11} + \frac{1}{1-p'} \log \alpha_{\varphi_1, p'_1}^*(\mathbf{C}'_{11}) \\ & \quad + \frac{1}{1-p'} \log \varpi_{n'_2}^*(p'_2, p') + \frac{1}{2} \log \det \mathbf{C}'_{22} + \frac{1}{1-p'} \log \alpha_{\varphi_2, p'_2}^*(\mathbf{C}'_{22}). \end{aligned}$$

Finally, after direct computations and inserting the constant  $\zeta(p', p'_1, p'_2)$ , the property claimed in (3.19) is obtained. ■

As closing, the inequality (3.19) is analyzed for particular case of  $\mathbf{C}$  and  $\mathbf{B}$ . So we offer

**Corollary 3.2** *Let  $\mathbf{Y}_1^*$  and  $\mathbf{Y}_2^*$  be  $n'_1$  and  $n'_2$  RVs, following Pearson's type VII with parameters  $\frac{p'}{1-p'_1}$  and  $\frac{p'}{1-p'_2}$  respectively. Further suppose that  $n' = n'_1 + n'_2$  RV  $\mathbf{Y}^{*T} = (\mathbf{Y}_1^{*T}, \mathbf{Y}_2^{*T})$  has Pearson's type VII with parameter  $\frac{p'}{1-p'}$  as well. Here  $p'$ ,  $p'_1$ ,  $p'_2$  are as in Theorem 3.2. Next let  $\underline{\lambda}^T = (\underline{\lambda}_1^T, \underline{\lambda}_2^T)$  be  $n'$  RV such that  $\underline{\lambda}_1^T = (\lambda_1, \dots, \lambda_{n'_1})$ ,  $\underline{\lambda}_2^T = (\lambda_{n'_1+1}, \dots, \lambda_{n'})$  and  $\lambda_i \geq 0$ . Consider  $\mathbf{\Lambda}_1$  and  $\mathbf{\Lambda}_2$  are the diagonal matrices with diagonal vectors  $\underline{\lambda}_1$  and  $\underline{\lambda}_2$  respectively, set*

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{pmatrix}.$$

Assume the following inequality is satisfied:

$$\int_{\mathbb{R}^{n'}} \prod_{i=1,2} |\mathbf{x}_i| g_{p'_i, 1}^{p'-1}(\mathbf{x}_i) \left[ g_{p', \mathbf{\Lambda}}(\mathbf{x}) - \prod_{i=1,2} g_{p'_i, \mathbf{\Lambda}_i}(\mathbf{x}_i) \right] d\mathbf{x} \leq 0, \quad \mathbf{x}_i \in \mathbb{R}^{n'_i}, i = 1, 2. \quad (3.20)$$

Then

$$\mathbb{E}^2 \left[ |\underline{\lambda}^T \mathbf{Y}^*| \right] \leq \eta(\underline{\lambda}) \mathbb{E}^2 \left[ |\underline{\lambda}_1^T \mathbf{Y}_1^*| \right] \mathbb{E}^2 \left[ |\underline{\lambda}_2^T \mathbf{Y}_2^*| \right], \quad (3.21)$$

where

$$\eta(\underline{\lambda}) = \left( \frac{\sum_{i=1}^{n'_1} \lambda_i^2}{\left( \sum_{i=1}^{n'_1} \lambda_i^2 \right) \left( \sum_{i=n'_1+1}^{n'} \lambda_i^2 \right)} \right)^{p'-1} \zeta(p', p'_1, p'_2) \left( \frac{2 \left( \frac{p'_1}{1-p'_1} - \frac{n'_1}{2} \right) \left( \frac{p'_2}{1-p'_2} - \frac{n'_2}{2} \right)}{\left( \frac{p'}{1-p'} - \frac{n'}{2} \right)} \right).$$

**Proof:** Suppose that  $\varphi_i(\mathbf{x}) = |\mathbf{x}|$ ,  $i = 1, 2$  The assertion (3.21) directly can be proved by choosing diagonal matrix  $\mathbf{B}$  with entries  $\lambda_i$ ,  $i = 1, \dots, n'$  and  $\mathbf{C} = \mathbf{I}$  is considered identity matrix in Theorem 3.2. ■

Invoking [9], for fixed  $p > 1$ , consider two  $n$ -dimensional RVs  $\mathbf{X}$ ,  $\mathbf{Y}$  with corresponding distributions  $g_{p, \mathbf{C}_\mathbf{X}}$ ,  $g_{p, \mathbf{C}_\mathbf{Y}}$ . Then the convolution  $\mathbf{X} *_p \mathbf{Y}$  defined below is also a Renyi entropy maximizer, having  $g_{p, \mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y}}$  PDF:

$$\mathbf{X} *_p \mathbf{Y} = \frac{U_\mathbf{X} \mathbf{X} + U_\mathbf{Y} \mathbf{Y}}{\sqrt{\left( U_\mathbf{X} \mathbf{X} + U_\mathbf{Y} \mathbf{Y} \right)^T (m \mathbf{C}_{\mathbf{X}\mathbf{Y}})^{-1} (U_\mathbf{X} \mathbf{X} + U_\mathbf{Y} \mathbf{Y}) + V}},$$

where  $\mathbf{C}_{\mathbf{X}\mathbf{Y}} = \mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y}$  and  $U_\mathbf{X}$ ,  $U_\mathbf{Y} \sim f_m$ ,  $m = n + 2p/(p-1)$  and  $V \sim f_m$  but  $m = 2p/(p-1)$  are independent random variables:

$$f_m(x) = \frac{2^{1-m/2}}{\Gamma(m/2)} x^{m-1} \exp\left(-\frac{x^2}{2}\right), \quad x > 0.$$

It is worthwhile to note that the convolution  $\mathbf{X} \circ \mathbf{Y}$  defined by

$$\mathbf{X} \circ \mathbf{Y} = \Theta_{(m-2)(\mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y})}^{-1} \left( \Theta_{(m-1)\mathbf{C}_\mathbf{X}}(\mathbf{X}) *_p \Theta_{(m-2)\mathbf{C}_\mathbf{Y}}(\mathbf{Y}) \right),$$

with  $m = 2/(1-p) - n$  and  $\tilde{p}$  satisfies in  $1/(\tilde{p}-1) = m/2 - 1$  and

$$\Theta_\mathbf{D}^{-1}(\mathbf{X}) = \frac{\mathbf{X}}{\sqrt{1 - \mathbf{X}^T \mathbf{D}^{-1} \mathbf{X}}},$$

follows distribution  $g_{p, \mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y}}$ . The above properties promoted us focus on  $g_{p, \mathbf{C}_\mathbf{X}}$  and  $g_{\mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y}}$  which is encapsulated in the following theorem. Before, consider a WF  $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \varphi(\mathbf{x}, \mathbf{y}) \geq 0$  and set

$$\varphi_{\mathbf{C}_\mathbf{X}}^*(\mathbf{x}) = \varphi(\mathbf{x}) \mathbf{x}^T \mathbf{C}_\mathbf{X}^{-1} \mathbf{x}.$$

Note that here we have

$$\mathbb{E}_{g_{p, \mathbf{C}_\mathbf{X}}}[\varphi_{\mathbf{C}_\mathbf{X}}^*] = \text{tr } \mathbf{C}_\mathbf{X}^{-1} \Psi_{\mathbf{C}_\mathbf{X}}^g \quad \text{and matrix } \Psi_{\mathbf{C}_\mathbf{X}}^g = \left( \psi_{ij}^{\mathbf{C}_\mathbf{X}} \right). \quad (3.22)$$

where

$$\psi_{ij}^{\mathbf{C}_\mathbf{X}} = \int_{\mathbb{R}^n} \varphi(\mathbf{x}) x_i x_j g_{p, \mathbf{C}_\mathbf{X}}(\mathbf{x}) \, d\mathbf{x}.$$

Moreover, similarly

$$\mathbb{E}_{g_{p, \mathbf{C}_\mathbf{X}}}[\varphi_{\mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y}}^*] = \text{tr } (\mathbf{C}_\mathbf{X} + \mathbf{C}_\mathbf{Y})^{-1} \Psi_{\mathbf{C}_\mathbf{X}}^g. \quad (3.23)$$

Now, owing to (3.22), (3.23) we provide a general result regarding the matrices, as stated by the following theorem.

**Theorem 3.3** *Let  $\mathbf{A}$ ,  $\mathbf{B}$  be two  $n \times n$  matrices. For given WF  $\varphi$  assume the following inequality involving  $g_{p, \mathbf{A}}$ :*

$$\begin{aligned} & \left( \det(\mathbf{A} + \mathbf{B}) \right)^{(1-p)/2} \left\{ \mathbb{E}_{g_{p, \mathbf{A}}}[\varphi] + (1-p)\beta \mathbb{E}_{g_{p, \mathbf{A}}}[\varphi_{\mathbf{A} + \mathbf{B}}^*] \right\} \\ & \leq (\geq) \left( \det \mathbf{A} \right)^{(1-p)/2} \left\{ \mathbb{E}_{g_{p, \mathbf{A}}}[\varphi] + (1-p)\beta \mathbb{E}_{g_{p, \mathbf{A}}}[\varphi_{\mathbf{A}}^*] \right\}, \quad p < 1 \, (p > 1). \end{aligned} \quad (3.24)$$

Then

$$\left( \frac{\alpha_{\varphi, p}^*(\mathbf{A} + \mathbf{B})}{\alpha_{\varphi, p}^*(\mathbf{A})} \right)^{1/(1-p)} \geq \left( \frac{\det(\mathbf{A} + \mathbf{B})}{\det \mathbf{A}} \right)^{1/2}, \quad p < 1. \quad (3.25)$$

For case  $p > 1$  substitute  $\alpha_{\varphi, p}$  in  $\alpha_{\varphi, p}^*$ .

**Proof:** We use the Renyi maximizing distributions with covariance matrices  $\mathbf{A}$  and  $\mathbf{A} + \mathbf{B}$ , i.e.  $g_{p,\mathbf{A}}$ ,  $g_{p,\mathbf{A}+\mathbf{B}}$ . Applying some straightforward computations, it can be seen that (3.24) implies

$$\frac{1}{1-p} \log \left( \int_{\mathbb{R}^n} \varphi g_{p,\mathbf{A}+\mathbf{B}}^{p-1} g_{p,\mathbf{A}} d\mathbf{x} \right) \leq \frac{1}{1-p} \log \left( \int_{\mathbb{R}^n} \varphi g_{p,\mathbf{A}}^p d\mathbf{x} \right).$$

By virtue of (2.7), we can write

$$h_{\varphi,p}^w(g_{p,\mathbf{A}+\mathbf{B}}) \leq h_{\varphi,p}^w(g_{p,\mathbf{A}}),$$

which consequently by inserting (1.16) for  $p < 1$  and (1.17) when  $p > 1$ , we obtain (3.25).  $\blacksquare$

**Remark:** Recalling the Sherman-Morrison formula (see [8], p. 161 and [18]), If  $\mathbf{A}$  and  $\mathbf{A} + \mathbf{B}$  be nonsingular matrices where  $\mathbf{B}$  is a matrix of rank one. Let  $\kappa = \text{tr}(\mathbf{B}\mathbf{A}^{-1})$ ,  $\mathbf{B}_{\mathbf{A}} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$ . Then  $\kappa \neq -1$  and condition (3.24) turns into the following inequality:

$$\begin{aligned} & \left( \left( \det(\mathbf{A} + \mathbf{B}) \right)^{(1-p)/2} - \left( \det \mathbf{A} \right)^{(1-p)/2} \right) \left( (1-p)\beta + 1 \right) \mathbb{E}_{g_{p,\mathbf{A}}}[\varphi] \\ & \leq (\geq) \frac{(1-p)\beta}{1+\kappa} \left( \det(\mathbf{A} + \mathbf{B}) \right)^{(1-p)/2} \mathbb{E}_{g_{p,\mathbf{A}}}[\varphi_{\mathbf{B}_{\mathbf{A}}}^*], \quad p < 1 \ (p > 1). \end{aligned}$$

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## References

- [1] S. M. Ali and S. D. Silvey. A general class of coefficients of divergence of one distribution from another. *J. Roy. Statist. Soc. Ser. B*, **28** (1966), 131–142.
- [2] M. Belis and S. Guiasu. A Quantitative and qualitative measure of information in cybernetic systems. *IEEE Trans. on Inf. Theory*, **14** (1968), 593–594.
- [3] A. Clim. Weighted entropy with application. *Analele Universității București, Matematică*, Anul **LVII** (2008), 223–231.
- [4] J. A. Costa, A. O. Hero and C. Vignat. A characterization of the multivariate distributions maximizing renyi entropy. *In proceedings of 2002 IEEE International Symposium on Information Theory*, (2002), page 263.
- [5] T. Cover and J. Thomas. *Elements of Information Theory*. New York: Wiley, 2006.
- [6] I. Csiszár. Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.*, **2** (1967), 299–318.
- [7] S. Guiasu. Weighted entropy. *Report on Math. Physics*, **2** (1971), 165–179.
- [8] G. Dahlquist and A. Björck, *Numerical Methods*, Prentice-Hall, Englewood Cliffs, NJ, 1974, p. 161).

- [9] O. Johnson and Ch. Vignat. Some results concerning maximum Rényi entropy distributions. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, **Vol. 43** (2007), 339–351.
- [10] J. N. Kapur. Generalised Cauchy and Student's distributions as maximum entropy distributions. *Proc. Nat. Acad. Sci. India Sect. A*, **Vol. 58** (1988), no. 2, 235–246.
- [11] M. Kelbert and Y. Suhov. *Information Theory and Coding by Example*. Cambridge: Cambridge University Press, 2013.
- [12] R. J. McEliece, E. R. Rodemich and L. Swanson. An entropy maximization problem related to optimal communication. *IEEE Trans. on Inform. Theory*, **Vol. 32** (1986), 322–325.
- [13] A. Renyi. On measures of entropy and information. In *Proc. Fourth Berkeley Symp. Math. Stat. Prob.*, **Vol. 1** (1960), page 547. Berkeley, (1961). University of California Press.
- [14] D. K. Song, H. J. Park and H. M. Kim. A note on the characteristic function of multivariate  $t$  distribution. *Communication for Statistical Application and Methods*, **Vol. 21** (2014), no. 1, 81–91.
- [15] A. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Inform. Contr.*, **Vol. 2** (1959), 101–102.
- [16] Y. Suhov, I. Stuhl, S. Yasaei Sekeh and M. Kelbert. Basic inequalities for weighted entropies. arXiv 1510.02184.
- [17] Y. Suhov, S. Yasaei Sekeh. An extension of the Ky Fan inequality. *arXiv:1504.01166*
- [18] Y. Suhov, S. Yasaei Sekeh and I. Stuhl. Weighted Gaussian entropy and determinant inequalities entropy. *arXiv:1502.02188*
- [19] Y. Suhov, S. Yasaei Sekeh and M. Kelbert. Entropy-power inequality for weighted entropy. arXiv: 1502.02188.
- [20] Y. Suhov, I. Stuhl, M. Kelbert. Weight functions and log-optimal investment portfolios. *arXiv:1505.01437*
- [21] S. Yasaei Sekeh. Extended inequalities for weighted Renyi entropy involving generalized Gaussian densities. *arXiv:1509.02190*
- [22] C. Vignat, J. Costa and A. O. Hero. Characterization of the multivariate distributions maximizing Tsallis entropy under covariance constraint. *Technical report, January 2003*, 214, 216, 218.
- [23] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge: Cambridge University Press, 1966.
- [24] R. Zamir. A proof of the Fisher information inequality via a data processing argument. *IEEE Transaction on Information Theory*, **44**, No. 3 (1998), 1246–1250.
- [25] K. Zografos. On maximum entropy characterization of Pearson's type II and VII multivariate distributions. *Journal of Multivariate Analysis*, **71**, No. 1 (1999), 67–75.



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